Linear Algebra Refresher

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Notations

Vector - We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the i^{th} entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

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Matrix - We note $A \in \mathbb{R}^{m \times n}$ a matrix with m rows and n columns, where $A_{i,j} \in \mathbb{R}$ is the entry located in the i^{th} row and j^{th} column

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

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Identity matrix - The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix with ones in its diagonal and zero everywhere else

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

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Diagonal matrix - A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is a square matrix with nonzero values in its diagonal and zero everywhere else:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

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- inner product: for $x, y \in \mathbb{R}^n$, we have:

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- outer product: for $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we have:

$$xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}$$

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Matrix-vector multiplication - The product of matrix $A \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$ is a vector y of size $\mathbb{R}^m$, such that for all $i \in \{1, \ldots, m\}$:

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Matrix-matrix multiplication - The product of matrix $A \in \mathbb{R}^{m\times n}$ and matrix $B \in \mathbb{R}^{n\times p}$ is a matrix C of size $\mathbb{R}^{m\times p}$, such that for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, p\}$:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

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Transpose - The transpose of a matrix is an operator which flips a matrix over its diagonal. Formally, the transpose of a matrix $A \in \mathbb{R}^{m \times n}$, noted A^T, is defined, for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, n\}$, as

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Inverse - $A \in \mathbb{R}^{n \times n}$ is said to be invertible if there exists $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I_n$. If this is the case, then the matrix B is uniquely determined by A, and is called the inverse of A, denoted A^{-1}.

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Determinant - The determinant of a matrix A, commonly denoted $\det(A)$, characterizes some properties of the matrix. In particular, the determinant is nonzero if and only if the matrix is invertible.

Results

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For matrices A, B, we have

$$(AB)^T = B^T A^T.$$

For square matrices A, B, we have

$$\det(A^T) = \det(A) \quad \text{and} \quad \det(AB) = \det(A)\det(B).$$